COMPUTER-ASSISTED EYE EXAMINATIONS IV. "ADDITIVE" LENS SYSTEMS IN EYE REFRACTORS*

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ABSTRACT

After a brief introduction to the optical principles and terms, formulas are derived which demonstrate that with conventional methods and designs it is not possible to summate with accuracy the labeled power of three or more ophthalmic lenses in tandem. The effective power of sphero-cylindrical combinations in all currently available eye refractors, when powers of eight to ten diopters are exceeded, may differ from the summated labeled power by a quarter diopter or more, which can be clinically significant. This error can be directly and readily eliminated only in computer-actuated refractors by including a computer program which calculates the effective power and makes the necessary correction.

Every student matriculating through a school of optometry learns about the effectivity of lenses. The presbyope can effectively change the power of his spectacles by sliding them along his nose towards or away from his eyes. Clearly the position of the lens can be an important factor in the sharpness of the retinal image.

Similarly, when trial lenses are placed before an eye, the power of each lens in effect depends on its distance from the eye. By convention the standard distance or spectacle plane is assumed to be 13 or 14 mm in front of the corneal apex. But when more than one lens is placed before an eye, only one of them can be in this plane at

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one time. This is true not only of trial lenses in a trial frame but also of the lenses in eye refractors.

How do the designers and manufacturers of eye refractors cope with this problem? What optical principles, if any, are used to avoid errors in the labeled power of various lens combinations? Are the errors, if any, clinically significant, and if so can any new principles be applied to eliminate them? The answers to these questions will be developed in this paper.

OPHTHALMIC OPTICS PRINCIPLES

The focal power of a lens and its focal length are two ways of expressing information about its image-forming properties. One is the reciprocal of the other. The power of a spectacle lens may be defined in a variety of ways. All the definitions are related mathematically to one another and any one may be deduced from another, provided that the parameters of the lens are known. The *nominal* power of a spectacle lens is the algebraic addition of the two surface powers. The effect due to the thickness of the lens is ignored. The term *equivalent* power is more applicable to thick lenses in which neither the center thickness, the refractive index nor the surface powers of the lens are ignored. This term expresses the magnitude of the power and the position of a thin lens which has equivalent paraxial image-forming properties. Modern spectacle lenses are usually of meniscus or toric form, and the equivalent focal lengths must be measured from the principal points. The positions of these points usually lie outside and in front of the substance of the lens, and vary with the degree of bending. Thus, it is more convenient for the optometrist or ophthalmologist to express lens power in somewhat more tangible terms. Measurements are made from the front or back vertex of a lens to its first or second principal focus, respectively. These distances are referred to as the front and back vertex focal lengths. Their reciprocals define the magnitudes of the front and back vertex powers of the lens. For this reason it is convenient to express the power of a spectacle lens by its back *vertex* power in units of lens power which are called diopters.

Since the optometrist may wish to prescribe a spectacle lens for use in some other plane in front of the eye than that occupied by the lenses used during the eye examination, a useful method of defining lens power is by means of its *effective* power. This definition may be used to express the power of a lens in any plane of operation.

If D_1 is the power of the front surface and D_2 is the power of the back surface of a spectacle lens of mean refractive index, n, and center thickness, t, the equivalent power, D_E, the back vertex power D_V', the front vertex power, D_V, and the nominal power D_N of the lens in the paraxial region may be expressed respectively, in the following way:[1]

$$D_E = D_1 + D_2 - \frac{t}{n} D_1 D_2 \qquad (1)$$

$$D_V' = D_E/(1 - \frac{t}{n} D_1) \qquad (2)$$

$$D_V = D_E/(1 - \frac{t}{n} D_2) \qquad (3)$$

$$D_N = D_1 + D_2 \qquad (4)$$

If the center thickness of the lens is ignored because its effect on the power of the lens is not significant, equations (1), (2) and (3) stated above may all be represented by the simpler thin lens equation (4).

The effective power of a lens in a plane, d meters from the lens, may be defined by the following equation:

$$D_{eff} = D/(1 - dD) \qquad (5)$$

LENS POWER SUMMATION

Several attempts have been made to produce trial lens sets and eye refracting instruments in which the back vertex power of the system of lenses is obtained by a simple algebraic summation of the nominal powers of the independent lenses in the system.[2] Theoretically, this is possible in two ways. One way is by making all the lenses plano-concave or plano-convex with the plane surfaces of any pair of lenses in opposition. All the spherical lenses must be the same thickness regardless of power, and the separation between the two lenses must be a fixed value. This is the Kellner principle.[3] The other also requires a fixed separation, but the front surface power and center thickness of the rear or ocular lens must be the same for all lenses. However, the lenses need not have one flat surface. This is the Tillyer principle.[4]

The Kellner principle is really a special case of the Tillyer principle. Kellner and Tillyer both stated the need to maintain a constant separation between the rear surface of the front lens and the first surface of the ocular lens of a twin lens trial system. Tillyer showed that the back vertex power of such a system is unaffected by the form of the front lens as long as the first surface powers and center thicknesses of all the ocular lenses in the series are held constant. The essential difference between the two principles follows. Tillyer showed that the power of the first surface of the ocular lenses may assume a constant value of any suitable magnitude, whereas Kellner assigned to it a power of zero magnitude. The equations and arguments which follow apply equally to both principles. They are applied to the Tillyer principle in this paper only because it constitutes the general case.

We will show that if the stack of lenses used for the eye examination exceeds two in number, there is no simple additive system by means of which the absolute back vertex power of the lens stack or its effective power in some other plane, for example, the spectacle plane, may be defined.

The Tillyer Principle

Tillyer[4] described a method by which two sets of lenses, A' and B', each containing a wide range of uniquely labeled lenses of different power, may be placed in tandem so that the back vertex power of any lens from set A' in combination with any lens from set B' is invariably equal to the algebraic sum of their labeled powers.

Formulation

Let two lenses of labeled powers A and B be placed in tandem so that the back vertex of lens A is separated from the front vertex of lens B by a distance t_1 (Figure 1). The back vertex

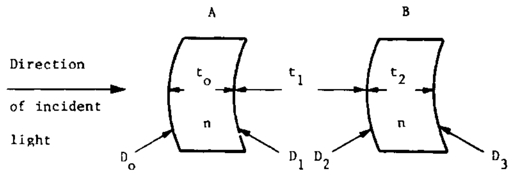

FIGURE 1: For an explanation, see text.

power of lens A is denoted by D'_{V_1}, the front surface power, back surface power and thickness of lens B are denoted respectively by D_2, D_3 and t_2. The refractive index of lenses A and B is denoted by n.

The back vertex power (D'_V) of the combination may be expressed as follows:

$$D'_V = \frac{D'_{V_1} + D_2 - t_1 D'_{V_1} D_2}{1 - t_1 D'_{V_1} - \frac{t_2}{n}(D'_{V_1} + D_2) + \frac{t_1 t_2}{n} D'_{V_1} D_2} + D_3 \qquad (6)$$

The labeled power B and the back vertex power of the second lens of the combination are equal and may be expressed as follows:

$$B = \frac{D_2}{1 - \frac{t_2 D_2}{n}} + D_3 \qquad (7)$$

Therefore the labeled power of lens A may be expressed by the equation:

$$A = D'_V - B = \frac{D'_{V_1} + D_2 - t_1 D'_{V_1} D_2}{1 - t_1 D'_{V_1} - \frac{t_2 (D'_{V_1} + D_2)}{n} + \frac{t_1 t_2 D'_{V_1} D_2}{n}} - \frac{D_2}{1 - \frac{t_2 D_2}{n}} \qquad (8)$$

B is the back vertex power of the second lens and A is the effective power of the first lens at the back vertex of the second lens. The labeled power of A is dependent only upon its back vertex power, its distance from B and the thickness, refractive index and front surface power of B. It is entirely independent of the surface powers and thickness of A and the back surface power of B. In such a system, the magnitude of D_2 and t_2 must be held constant for all lenses in set B, but may be selected to optimize their image forming properties. D_3 is altered to provide each lens of set B with the correct back vertex power. Similarly, the surface powers and thicknesses of the lenses in set A may be selected to control aberrations. The back vertex power D'_{V_1} of the lenses in set A may be expressed as follows:

$$D'_{V_1} = \frac{D_0 + D_1 - \frac{t_0 D_0 D_1}{n}}{1 - \frac{t_0 D_0}{n}} \qquad (9)$$

where D_0 and D_1 are the surface powers of A, and t_0 and n are its center thickness and refractive index, respectively.

Tillyer stated categorically that the principle which he enunciated for two lenses may be applied to a combination of any number of lenses selected from similar sets. A large number of so-called "Additive" phoropter heads or eye refractors have been manufactured on the validity of Tillyer's claim for the algebraic additivity of combinations of more than two lenses in series. There are no theoretical grounds upon which this claim can be substantiated.

In most routine techniques for evaluating the refractive condition of the human eye, often more than two lenses are used in series. Since these lenses are placed either in the cells of a trial frame or in the disks of an eye refractor, an air space is invariably present between consecutive lenses. Furthermore, the lenses of higher power cannot be treated as thin lenses. If the independent lenses of a combination containing more than two lenses are each labeled with a unique power function, there are no theoretical grounds for assuming that the algebraic sum of the labeled powers will equal the back vertex power of the system. This problem of lens effectivity in the use of trial lenses and refractors for investigating the refractive state of the eye has been realized by the manufacturers of these instruments since at least the turn of the century. The various unsuccessful attempts which have been made to solve it have included systems in which the positions of the principal and nodal points have been held stationary, and adjustments to the actual power of a lens used in a prescribed plane, so that the correct effective power will be provided in some other plane. Attempts have been made to minimize effectivity errors by locating the high powered spherical lenses closest to the patient's eye and reducing the air space between successive lenses to a minimum. However, the cylindrical lenses sometimes occupy positions in the sight hole of the instrument which are in excess of 15 mm in front of the reference plane containing the disk of high powered spherical lenses. Cumbersome, bulky lens systems with surface powers that produce undesirable aberrations are often the result of the first approach to the problem. The second technique is perfectly valid if either one lens only is employed or if the reference plane coincides with the back vertex of the second lens of a two-lens system. However, this method of simple additive refraction will almost invariably fail for a system of more than two lenses in series, even though each one of them may provide the correct effective power independently.

Formulation

Consider the simple case of three thin lenses of powers D_1, D_2 and D_3 separated by distance t_1 and t_2 respectively (Figure 2).

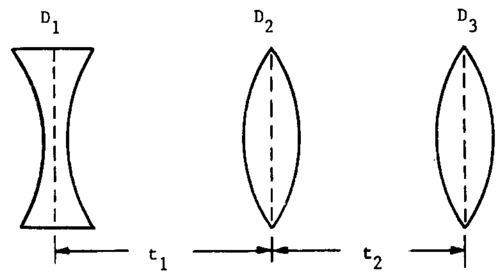

FIGURE 2: For an explanation, see text.

The effective powers of each of the lenses in the plane of D_3 are as follows:

$$D_{1_{EFF}} = \frac{D_1}{1 - D_1(t_1 + t_2)} \qquad (10)$$

$$D_{2_{EFF}} = \frac{D_2}{1 - t_2 D_2} \qquad (11)$$

$$D_{3_{EFF}} = D_3 \qquad (12)$$

The sum of the effective power ΣD_{EFF} in the plane of D_3 is given by equation (13).

$$\Sigma D_{EFF} = \frac{D_1 + D_2 - D_1D_2(t_1 + 2t_2)}{1 - t_1D_1 - t_2D_1 - t_2D_2 + t_1t_2D_1D_2 + t_2^2D_1D_2} + D_3 \quad (13)$$

On the other hand, the back vertex power D_V' of the system may be expressed by the equation (14).

$$D_V' = \frac{D_1 + D_2 - t_1D_1D_2}{1 - t_1D_1 - t_2D_1 - t_2D_2 + t_1t_2D_1D_2} + D_3 \quad (14)$$

Equations (13) and (14) are not equal unless t_2 and/or D_2 are equal to zero. Either action effectively converts the three lens systems into a two lens system.

Now consider the case of three thick lenses of labeled powers A, B and C which are independent members of three sets of lenses A', B' and C', respectively. All lenses have a common refractive index n. All lenses in set A' are of constant center thickness, t_1, while those in set B' are of constant center thickness t_3 and those in set C' are of constant center thickness, t_5. The front surface power of all lenses in set A' is D_1 and that of all lenses in sets B' and C' is D_3 and D_5 respectively. The lenses of set A' are separated from those of set B' by an air space of constant center thickness t_2 and the lenses of set B' are separated from those of set C' by an air space of constant center thickness t_4 (Figure 3).

The back vertex powers of lenses A, B and C are D_{V_1}', D_{V_2}' and D_{V_3}' respectively. Tillyer claimed that the back vertex power of the system, D_V', is always equal to the algebraic sum of any three etc. labeled powers A, B and C etc.

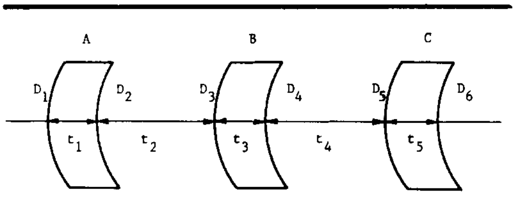

FIGURE 3: For an explanation, see text.

taken from lens-sets A', B' and C' etc.

$$D_{V_3}' = C = \frac{D_5}{1 - \frac{t_5D_5}{n}} + D_6 \quad (15)$$

$$\frac{D_5}{1 - \frac{t_5D_5}{n}} \text{ is a constant.}$$

$$D_6 - C = -\left[\frac{D_5}{1 - \frac{t_5D_5}{n}}\right] = \text{constant.} \quad (16)$$

If a second lens of labeled power B is added from set B', then the back surface power of this lens may be stated as follows:

$$D_4 = \frac{D_3 + D_5 - D_3D_5(\frac{t_3}{n} + t_4) + \left\{1 - \frac{t_5D_5}{n}\left[1 - D_3(\frac{t_3}{n} + t_4)\right] - D_3(\frac{t_3}{n} + t_4 + \frac{t_5}{n})\right\} \cdot (D_6 - B - C)}{(D_6 - B - C)\left[t_4 + \frac{t_5}{n} - \frac{D_5t_4t_5}{n}(1 - \frac{t_3}{n}D_3)\right] + (t_4D_5 - 1)(1 - \frac{t_3D_3}{n})} \quad (17)$$

For a two lens system, the magnitude of D_4 is dependent only upon the magnitude of B, the remaining terms of the equation being constant. Thus, every lens B in the lens set B' may be assigned a unique back surface power. Similarly, the back vertex power D_{V_2}' of lens B is dependent only upon its labeled power.

$$D_{V_2}' = \frac{(D_6 - B - C)(1 - \frac{t_5D_5}{n}) + D_5}{\frac{t_5}{n}(D_6 - B - C) + t_4(D_6 - B - C)(1 - \frac{t_5D_5}{n}) + t_4D_5 - 1} \quad (18)$$

Once again every lens B in the lens set B' may be assigned a unique back vertex power.

However, if a third lens of labeled power A is added from set A', the required back surface power RD_4 of lens B and the power of the system D'_{s_2} at this point are dependent on the labeled powers of both A and B. Thus, for the three lens system:

$$RD_4' = \frac{D_3+D_5-D_3D_5(\frac{t_3}{n}+t_4) + \{ 1 - \frac{t_5D_5}{n} \; [1 - D_3(\frac{t_3}{n}+t_4)] - D_3(\frac{t_3}{n} + t_4 + \frac{t_5}{n}) \} \cdot (D_6-A-B-C)}{(D_6-A-B-C) \; [t_4 + \frac{t_5}{n} - \frac{D_5t_4t_5}{n} (1 - \frac{t_3D_3}{n})] + (t_4D_5-1)(1 - \frac{t_3D_3}{n})} \qquad (19)$$

$$D'_{s_2} = \frac{(D_6 - A - B - C)(1 - \frac{t_5D_5}{n}) + D_5}{\frac{t_5}{n}(D_6 - A - B - C) + t_4(D_6 - A - B - C)(1 - \frac{t_5D_5}{n}) + t_4D_5 - 1} \qquad (20)$$

Equation (19) implies that for a given lens B from set B' the magnitude of RD_4 should be changed every time lens A is changed. Such a system is absurd. Therefore, the back vertex power of a set of more than two lenses should not be stated as a simple algebraic addition of their labeled powers.

Since the back surface power and back vertex power of each lens in set B' is unique, the lenses of set A' will not always provide their labeled effect at the back vertex of the system. Each lens A will provide an effect which varies as the lenses of set B' are exchanged. This conclusion is supported by equation (22).

The back vertex power D'_{v_1} of lens A in a system of three separated lenses A, B and C may be stated in terms of its own parameters by equation (21). The required back vertex power D'_{s_1} of a lens A to provide its labeled effect A, at the back vertex of the system, may be stated in terms of the parameters of lenses B and C by equation (22).

$$D'_{v_1} = \frac{D_1 + D_2 - \frac{t_1D_1D_2}{n}}{1 - \frac{t_1D_1}{n}} \qquad (21)$$

See below for (22)

Reference to equation (22) also shows that the required back vertex power of lens A is dependent upon the back surface power D_4 of lens B. Since D_3 and t_3 are constant for all the lenses of set B' and the magnitude of D_4 is a function of the magnitude of B for each of the lenses of this set, the labeled effect of lens A will vary as the lenses of set B' are interchanged.

Trivial Forms

On the assumption that the three lenses are thin and their separation distances are negligible, equations (17) to (22) reduce to the following trivial form.

$$D_4 = B - D_3 \qquad (17A)$$

$$D'_{v_2} = B \qquad (18A)$$

$$RD_4 = A + B - D_3 \qquad (19A)$$

$$D'_{s_2} = A + B \qquad (20A)$$

$$D'_{s_1} = A \qquad (22A)$$

$$D'_{s_1} = \frac{D_3+D_4+D_5+(D_6-A-B-C) \cdot \{ (1 - \frac{t_5D_5}{n}) [(1-t_4D_4)(1 - \frac{t_3D_3}{n}) - t_4D_3] - \frac{t_5}{n} \cdot (D_3+D_4 - \frac{t_3D_3D_4}{n}) \} - t_4D_5(D_3+D_4) - \frac{t_3D_3}{n}(D_4+D_5-t_4D_4D_5)}{(D_6-A-B-C) \cdot \{ 1 - \frac{t_5D_5}{n}) \cdot [t_2 + \frac{t_3}{n} + t_4 - t_4D_4(t_2 + \frac{t_3}{n} - \frac{t_2t_3D_3}{n}) - t_2D_3(\frac{t_3}{n} + t_4)] + \frac{t_5}{n}(1-t_2D_3) - \frac{t_5D_4}{n}(t_2 + \frac{t_3}{n} - \frac{t_2t_3D_3}{n}) \} - (1 - t_4D_5) \cdot \{ 1-t_2D_3-D_4(t_2 + \frac{t_3}{n} - \frac{t_2t_3D_3}{n}) \} + D_5(t_2 + \frac{t_3}{n} - \frac{t_2t_3D_3}{n})} \qquad (22)$$

However, it is clear from equations (17A) and (19A) that the required back surface power of D_4 (RD_4) of lens B and its actual back surface power (D_4) are equal only if the power of lens A is zero.

Since the magnitudes of t_1 to t_5 are each of the order of a few millimeters, terms containing the product of t may be neglected without significantly reducing the accuracy of the equations. If this approximation is made, a number of terms in t are still retained in the above equations. The remaining t terms are of the same order of magnitude as those which Tillyer took into account in the design of his additive system. The inclusion of these items in a system of more than two lenses may produce a significant difference between the actual effectivity of the system and its labeled effectivity. That is, a difference in excess of 0.25 diopters in the case of some of the high-powered spherocylindrical combinations.

PRACTICAL CONSIDERATIONS

If the algebraic sum of the labeled lens powers in the stack is relatively low, there is often no significant difference between it and the true back vertex power. On the other hand, if the algebraic sum of the stack labels exceeds about 8.00 or 10.00 diopters, significant differences frequently do exist and some compensation must be made.

Even if there is good agreement between the nominal prescription displayed on the indicators of the eye refractor and the true back vertex of the lens stack, this prescription will only be duplicated by a spectacle lens of similar back vertex power if the back vertex of the latter occupies a position in front of the patient's eye similar to that formerly occupied by the back vertex of the lens stack.

This can seldom be achieved with an eye refractor. The back vertex of the main lens stack is usually in excess of 7 mm in front of the face plate of the instrument, while the auxiliary disk containing the occluder is only about 2.5 mm in front of the face plate. The patient's eyelashes must clear this occluder. If an allowance of 2 mm is made for the thickness of the upper eyelid, a vertex distance of 12 mm can be maintained only if the upper eyelashes are less than about 5.5 mm in length. Since adult upper eyelashes generally average 7 mm in length and a clearance of at least 1 mm between the lashes and the occluder is desirable, the minimum distance from the vertex of the lens stack at which the patient's cornea may be conveniently located is about 14 mm. In some eye refractors in which the sight hole is fitted with protective glass cover, the vertex distance/effectivity problem is even more acute.

In general, patients do not feel comfortable when placed so close to an eye refractor. There are a number of reasons for this. If the patient is placed too close to the instrument, ventilation of the lenses may be impaired and they may become moisture fogged. If the patient's eyelashes have made involuntary contact with some part of the auxiliary disk during its movement, the patient will make a reflex movement away from the instrument. If a rigid headrest is used to prevent this backward movement of the head, the patient may become disturbed and tense by the rigid immobility which has been imposed upon him. Because the instrument may obstruct the practitioner's view of the patient, he may not always be aware that the patient has moved.

An improved method of maintaining a constant back vertex distance and of monitoring this distance, and a simple method of calculating the actual spectacle correction in any desired plane is, therefore, required. This may be achieved in a number of ways. Small proximity or microswitches may be fitted to the headrest and operated by contact with the patient's forehead so that a warning light or buzzer is activated immediately if the patient moves out of position. Some eye refractors are fitted with alignment devices with which to adjust and measure the back vertex distance. Others are not equipped with such devices and would be improved by the inclusion of a simple alignment and monitoring device of this kind. If there is a mismatch between the vertex distance of the spectacle lens and that of the eye refractor, it may be necessary for the practitioner to take into account this difference in calculating the actual spectacle prescription. The practitioner could make use of a small programmable desktop calculating machine or a dedicated microcomputer to calculate the power of the spectacle lens required, taking into account not only variations in back vertex distance, but also the form, thickness and separation of the individual lenses in the lens stack of the eye refractor. Such a calculator may also be used to compute the most suitable form of spectacle lens for the patient. Frequently the practitioner does none of these things but often resorts to very empirical and less satisfactory methods of writing the final prescription. He may obtain the best results that he can with the aid of the eye refractor, transfer the prescription to a trial frame for modification and refinement and finally measure the back vertex power of the trial lenses with a focimeter or lensometer. Such empirical approaches to ocular refraction are time consuming, expensive, inefficient and often far from accurate.

In the case of the computer-actuated eye refractor[5] such problems need never arise and are easily dealt with by the computer itself. To limit the range of this instrument is unnecessary. The system has been programmed to take into account variations in effectivity of the lens stack and print out the required prescription at the actual spectacle plane of the patient.

Two different methods can be used to correct for effectivity.[a] In the first or final correction all the tests are performed using nominal refractive values. At the conclusion of the tests the final recommended prescription would be calculated for effective value and printed out as such. The final correction method requires no extra computation during testing and it is not complex, but uses a simple algorithmic method. However, the steps of power are not uniform, some being more and others less than the assumed step, usually a quarter diopter.

The second or continuous correction method is to have the computer calculate which combination of lenses gives each desired effective power as used during testing and then choose them. Although the method is ideal for the examination, the computations are very complex. It becomes a trade-off between the availability or cost of extra computation capacity and the cost of the additional time testing might take because of the lack of equal increments of testing power. In principle, the final correction method may not be quite as accurate as the continuous method, but practically, there may be little difference in the results of the two methods.

A combination of the two methods can be used by choosing an approximately correct effective power and then calculating the necessary correction for complete accuracy. This would require more computation than the final method, but less than the continuous method.

Such elegance and accuracy in the solution of a fundamental problem of eye refractive examinations is possible only with computer operated systems.

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[a]These methods were evolved in discussions with Dr. Chacko Neroth.